General conditions for a quantum adiabatic evolution

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The smallness of the variation rate of the hamiltonian matrix elements compared to the (square of the) energy spectrum gap is usually believed to be the key parameter for a quantum adiabatic evolution. However it is only perturbatively valid for scaled timed hamiltonian and resonance processes as well as off resonance possible constructive Stückelberg interference effects violate this usual condition for general hamiltonian. More general adiabatic condition and exact bounds for adiabatic quantum evolution are derived and studied in the framework of a two-level system. The usual criterion is restored for real two level hamiltonian with small number of monotonicity changes of the hamiltonian matrix elements and its derivative.

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Adiabaticity is at the border between dynamics and statics. It has been introduced by Boltzmann in classical mechanics and by Born and Fock in 1928 in Quantum Mechanics [1, 2], extended to the infinite dimensional setting by Kato (1950), studied as a geometrical holonomy evolution by Berry (1984), finally extended to degenerate cases (without gap condition) and to open quantum system more recently [3, 4]. The quantum adiabatic theorem is usually used to derive approximate solutions of the Schrödinger equation and is strongly related to the (semi-)classical limit $\hbar \to 0$ of quantum mechanics [5] and to the Minimal work principle [6] for the Hamiltonian $H(t)$. The principle is simple: if a quantum system is prepared in an eigenstate $|n(0)\rangle$ of a "slowly" varying Hamiltonian it remains (without taking into account of the phase evolution) close to the instantaneous eigenstate $|n(t)\rangle$ of this Hamiltonian as time t goes on. The applications range from two-level systems (nuclear magnetic resonance, atomic laser transitions, Born-Oppenheimer molecular adiabatic coupling, collisional processes ...) to quantum algorithms [7].

"Usual" adiabatic conditions are (for all $t \in [0, T]$):

$$\sum_{m\neq n} \frac{1}{|\omega_{mn}(t)|} \frac{|\dot{H}_{mn}(t)|}{|E_{mn}(t)|} = \sum_{m\neq n} \left| \frac{\langle m(t)|\dot{n}(t)\rangle}{\omega_{mn}(t)} \right| \ll 1, \quad (1)$$

where the dot designs the time derivative and $|m\rangle$ are the instantaneous eigenstates for the energy eigenvalue $E_m(t)$ with $E_{mn} = \hbar\omega_{mn} = E_m - E_n$ [27]. Some confusion occurs recently [8, 9, 10, 11, 12, 13] because, this condition seems written for a general hamiltonian $H(t)$. However, it has been studied by many different techniques (see for instance [14, 15]) but only for special types of hamiltonian such as time scaling one $H(t) = \hat{H}(t/T)$ [28]. Furthermore, even for such a time scaled hamiltonian, condition (1) is not sufficient because it is only the leading order term [16, 17], in a time evolution T perturbation point of view, and more accurate conditions are needed to prove adiabatic evolution [15].

The goal of this article is to derive general quantum adiabatic conditions for general hamiltonian. We start our study on a two level system example in order to study some possible violation of the usual adiabatic conditions. Afterwords, considering a more general type of N levels hamiltonians, we derive a general criterion for adiabaticity. Finally, the study of the interference during multiple passages allows us to precise the validity of the usual adiabatic condition.

A quite general 2×2 hamiltonian matrix, written in the Pauli Matrix ($\vec{\sigma}$) basis, leads to the a spin 1/2 form $H = -\hbar\frac{\gamma}{2}\vec{B}.\vec{\sigma}$:

$$H = -\frac{\hbar\omega_0}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} \delta_0+\omega_L & \Omega_0 e^{i\int\omega_L} \\ \Omega_0 e^{-i\int\omega_L} & -\delta_0-\omega_L \end{pmatrix}$$

where $\omega_0 = \gamma B$ is the Larmor frequency, $\vec{B}$ is a rotating magnetic field with a polar angle θ, an azimuthal rotating angular frequency $\dot{\varphi} = -\omega_L$. Where the second form of the hamiltonian represents, in the rotating wave approximation (RWA), a two level system coupled to an external (laser with angular frequency ω_L for instance) field which is frequency detuned by $\delta_0 = \omega_0\cos\theta - \omega_L$ from the resonance and with a real Rabi frequency $\Omega_0 = \omega_0 \sin\theta$. For future developments we also define $\Omega_L = \omega_L \sin\theta - i\dot{\theta} = |\Omega_L|e^{i\arg\Omega_L}$, $\delta_L = \omega_L\cos\theta - \omega_0 + \frac{d}{dt}\arg\Omega_L$ and $\Omega_R = \sqrt{|\Omega_L|^2 + \delta_L^2}$. The eigenvectors $e^{i\theta_\mp}|\mp\rangle$, corresponding to the eigenvalues $\mp\hbar\omega_0/2$, are given by the columns of $R_\theta = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\theta}{2}e^{i\theta_-} & -e^{-i\frac{\varphi}{2}}\sin\frac{\theta}{2}e^{i\theta_+} \\ e^{i\frac{\varphi}{2}}\sin\frac{\theta}{2}e^{i\theta_-} & e^{i\frac{\varphi}{2}}\cos\frac{\theta}{2}e^{i\theta_+} \end{pmatrix}$. The evolution of the amplitudes b_- and b_+ of the $e^{i\theta_-}|-\rangle$ and $e^{i\theta_+}|+\rangle$ states are driven by the hamiltonian $\tilde{H} = R_\theta^\dagger H R_\theta - i\hbar R_\theta^\dagger \dot{R}_\theta$:

$$\tilde{H} = \frac{\hbar}{2} \begin{pmatrix} \omega_L\cos\theta - \omega_0 + 2\dot{\theta}_- & -\Omega_L e^{i\theta_{+-}} \\ -\Omega_L^* e^{i\theta_{-+}} & \omega_0 - \omega_L\cos\theta + 2\dot{\theta}_+ \end{pmatrix}$$

with $\theta_{+-} = \theta_+ - \theta_- = -\theta_{-+}$. One natural choice for

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$\theta_\pm$ is the "first order" choice $\theta_\pm^{(1)} = \mp\frac{1}{2}\int_0^t \omega_0 - \omega_L\cos\theta$ annulling the whole diagonal terms.

Let us treat the (Schwinger 1937) example, where all the parameters $\omega_0, \theta, \dot{\varphi} = -\omega_L$ are real and time independent. The evolution operator in the adiabatic $|\mp\rangle$ basis $\tilde{U}(t,0) = R_\theta^\dagger(t)U(t,0)R_\theta(0)$ (where $U(t,0)$ is the evolution operator in the diabatic basis) verifies $i\hbar\dot{\tilde{U}} = \tilde{H}\tilde{U}$ and, with $\theta_\pm = \theta_\pm^{(1)}$, is given by the matrix:

$$\tilde{U} = \begin{pmatrix} (\cos\frac{\Omega_R t}{2} - i\frac{\delta_L}{\Omega_R}\sin\frac{\Omega_R t}{2})e^{i\frac{\delta_L t}{2}} & ie^{i\frac{\delta_L t}{2}}\frac{\Omega_L}{\Omega_R}\sin\frac{\Omega_R t}{2} \\ ie^{-i\frac{\delta_L t}{2}}\frac{\Omega_L}{\Omega_R}\sin\frac{\Omega_R t}{2} & (\cos\frac{\Omega_R t}{2} + i\frac{\delta_L}{\Omega_R}\sin\frac{\Omega_R t}{2})e^{-i\frac{\delta_L t}{2}} \end{pmatrix}$$

The adiabaticity (negligible off-diagonal terms in $\tilde{U}$) evolution is given by the following condition $A^{(2)} = \frac{|\Omega_L|}{\Omega_R} = \frac{|\omega_L\sin\theta|}{\sqrt{\omega_0^2 - 2\omega_L\omega_0\cos\theta + \omega_L^2}} \ll 1$, where here $\Omega_R = \sqrt{\Omega_0^2 + \delta_0^2}$ is the generalized Rabi frequency. Using $\Omega_R = \sqrt{|\Omega_L|^2 + \delta_L^2}$ this adiabatic condition can be written

$$2A^{(1)} = \left|\frac{\Omega_L}{\delta_L}\right| = \left|\frac{\omega_L\sin\theta}{\omega_0 - \omega_L\cos\theta}\right| \ll 1 \qquad (2)$$

which has to be compared with the "usual" adiabatic condition given by Eq. (1):

$$2A^{(0)} = \left|\frac{\Omega_L}{\omega_0}\right| = \left|\frac{\omega_L\sin\theta}{\omega_0}\right| \ll 1. \qquad (3)$$

$A^{(0)}, A^{(1)}, A^{(2)}$ notations will be generally defined latter.

Looking at the $\omega_L \approx \omega_0$ and θ very small resonant case ($\delta_L \approx \delta_0 \approx 0$), we see, in a simpler way than in Ref. [8, 9] and contrary to what is sometimes claimed [12, 13], that Eq. (3) is verified but not Eq. (2). This fundamental conclusion, based on a hamiltonian $H(t) \neq \hat{H}(t/T)$ is still valid for the time scaling case $\hat{H}(t/T)$. Indeed, the Schwinger hamiltonian can be of the $\hat{H}(t/T)$ type if $\omega_L T$ is taken to be constant, for instance by looking at the evolution after one period $T = T_L = 2\pi/\omega_L$ depending on the ω_L parameter value. Indeed, $2A^{(1)} = \frac{1}{T_L}\left|\frac{\sin\theta}{\omega_0} + O(T_L^{-2})\right| = 2A^{(0)}\left|1 + \frac{2A^{(0)}}{\tan\theta} + O(T_L^{-2})\right|$ indicates, for instance if θ is very small, why an evolution time T_L much longer than expected by the usual condition ($T_L \gg \left|\frac{\sin\theta}{\omega_0}\right|$) can be needed to provide adiabatic evolution. *The "usual" adiabatic conditions are then clearly not sufficient to provide adiabatic evolution even for $\hat{H}(t/T)$ hamiltonian type.*

To be more general let us now study a discrete, but possibly degenerate, hamiltonian with the state evolution $|\Psi(t)\rangle = \sum_{m=1}^N b_m(t)e^{i\theta_m(t)}|m(t)\rangle$ ($N \geq 2$). The phase $\theta_m = \gamma_m + \alpha_m$ is real but not necessary equals to the first order choice $\theta_m^{(1)} = \int_0^t i\langle m|\dot{m}\rangle - \int_0^t E_m/\hbar$: geometrical phase (which is the Berry Phase for cyclic evolution) plus dynamical phase neither contains the (Pancharatnam) phase $\arg\langle m(0)|m(t)\rangle$. To study the adiabatic evolution we shall assume that $|\Psi(t=0)\rangle = |n(0)\rangle$ (i.e. $b_n(0) = 1$). The evolution is adiabatic if $1 - |\langle n(T)|\Psi(T)\rangle| = 1 - |b_n(T)| \ll 1$ or equivalently if $\||\Psi\rangle\langle\Psi| - |n\rangle\langle n|\| = \sqrt{1 - |b_n^2|} \ll 1$ [15].

The Schrödinger's equation leads for each m state to:

$$\dot{b}_m = -ib_m\left(\dot{\theta}_m - \dot{\theta}_m^{(1)}\right) - \sum_{k\neq m} b_k\langle m|\dot{k}\rangle e^{i\theta_{km}} \qquad (4)$$

where $\theta_{km} = \theta_k - \theta_m$. Using $\dot{\theta}_m = \dot{\theta}_m^{(1)}$, $-\frac{d|b_n|}{dt} \leq \left|\frac{db_n}{dt}\right|$ and the norm inequality $\sqrt{N-1}\sqrt{1-|b_n^2|} = \sqrt{N-1}\sqrt{\sum_{m\neq n}|b_m|^2} \geq \sum_{m\neq n}|b_m|$ we find the first (very restrictive) valid adiabatic condition for the interaction time T:

$$1 - |b_n(T)| \leq 1 - \cos(\sqrt{N-1}\Omega_n T) \leq (N-1)\frac{\Omega_n^2}{2}T^2 \quad (5)$$

where $\Omega_n = \max_{t\in[0,T]}{}_{m\neq n}|\langle n(t)\,|\dot{m}(t)\,\rangle|$. This condition is optimal because it is reached (see $\tilde{U}$) by the Schwinger $N = 2$ level system for $\delta_L = 0$ ($\Omega_n = |\Omega_L| = \Omega_R$). It illustrates the quantum Zeno effect: during a time much smaller than $\frac{1}{\sqrt{N}\Omega_n}$ the system evolution is frozen.

In order to find more useful adiabatic conditions we integrate by part Eq. (4) using (for $k \neq m$) $A_{km} = \frac{\langle m|\dot{k}\rangle e^{i(\theta_{km}-\gamma_{km})}e^{i(\theta_k^{(1)}-\theta_k)}}{\dot{\gamma}_{km}}$:

$$\begin{aligned} b_m(T) - b_m(0) = &\sum_{k\neq m}\left[ib_k(t)e^{i\gamma_{km}(t)}e^{i(\theta_k(t)-\theta_k^{(1)}(t))}A_{km}(t)\right]_0^T \\ &-i\int_0^T b_m\left(\dot{\theta}_m - \dot{\theta}_m^{(1)} - \sum_{k\neq m} e^{i(\gamma_{km}+\theta_m-\theta_k^{(1)})}A_{km}\langle k|\dot{m}\rangle\right) \\ &-i\sum_{k\neq m}\int_0^T b_k e^{i\gamma_{km}}e^{i(\theta_k-\theta_k^{(1)})}\dot{A}_{km} \qquad (6) \\ &+i\sum_{k\neq m}\int_0^T b_k\sum_{j\neq k,m} e^{i(\gamma_{jm}+\theta_k-\theta_j^{(1)})}A_{jm}\langle j|\dot{k}\rangle \end{aligned}$$

It is now straightforward, with $m = n$, to look back to the standard adiabatic theorem with the time scaling $t = sT$. The evolution equation for $|\hat{\Psi}(s)\rangle = |\Psi(t(s))\rangle$, is then $i\frac{\hbar}{T}\frac{d}{ds}|\hat{\Psi}(s)\rangle = \hat{H}(s)|\hat{\Psi}(s)\rangle$ and the $T \to +\infty$ limit is similar to $\hbar \to 0$. With $\gamma_{km} = E_{mn}/\hbar$, we have (for $\theta_k = \theta_k^{(1)}$) $A_{km} = A_{km}^{(0)} = \frac{\langle m|\dot{k}\rangle e^{-\int(\langle k|\dot{k}\rangle - \langle m|\dot{m}\rangle)}}{\frac{E_m - E_k}{\hbar}}$ and the stationary phase theorem (saddle-point or steepest descent method) annuls, for $T \to +\infty$, the integrals in Eq. (6) leading to valid quantum adiabatic condition:

$$\sum_{m\neq n}\frac{1}{T}\left|\frac{\hbar\frac{d\hat{H}}{ds}\Big|_{mn}}{(E_{mn})^2}\right| + o\left(\frac{1}{T}\right) = \sum_{m\neq n}|A_{mn}^{(0)}| + o\left(\frac{1}{T}\right) \ll 1$$

A comparison with Eq. (1) indicates, as also shown by the two level model where $|A_{+-}^{(0)}| = \frac{|\Omega_L|}{2|\omega_0|}$, that a better

understanding of the $o\left(\frac{1}{T}\right)$ term is in fact needed to have useful condition [15].

We could now go back to the general $H(t)$ case. $\sqrt{1-|b_n(T)|^2}$ verifies $\sqrt{1-|b_n(T)|^2} \leq \sqrt{N-1}b_-$ with $b_- = \max_{t\in[0,T]} \max_{m\neq n} |b_m(t)|$. Using Eq. (6) and $\theta_m = \theta_m^{(1)}$ choice, it be bounded by

$$b_- \leq \frac{2A + \int|A'| + (N-2)A\Omega T}{1-(N-2)(A+\int|A'|) - ((\mathtt{N}-1) + (N-2)^2)A\Omega T}.$$

The typewriter style, such as $(\mathtt{N}-1)\mathtt{A\Omega T}$, indicates terms that can be annulled by using a better phase for θ_m namely the "second order" one $\theta_m^{(2)} = \theta_m^{(1)} + \int_0^t \sum_{k\neq m} e^{i(\gamma_{km}+\theta_m-\theta_k^{(1)})} A_{km}\langle k|\dot{m}\rangle$. The three important parameters are:

$$\begin{aligned}
\Omega &= \max_{\substack{t\in[0,T]\\ k\neq m}} |\langle m|\dot{k}\rangle| = \max_m \Omega_m \leq \max_{t\in[0,T]} \frac{\|\dot{H}\|}{\Delta E} \\
A &= A(T) = \max_{\substack{t\in[0,T]\\ k\neq m}} |A_{km}(t)| \leq \frac{\Omega}{\min\limits_{\substack{t\in[0,T]\\ k\neq m}} \dot{\gamma}_{km}} \\
\int|A'| &= \max_{k\neq m} \int_0^T \left|\dot{A}_{km}\right|
\end{aligned}$$

Where, $\Delta E = \min_{k\neq m} E_{km}$ is the energy spectrum gap. Another (better for large T) bound for $b_+(T) = \min_{t\in[0,T]} |b_n(t)| = |b_n(t_T)|$ is obtained using $m = n$ in Eq. (6) and the norm inequality:

$$\begin{aligned}
&1-|b_n(t_T)| \leq (\mathtt{N}-1)\mathtt{A\Omega T} \\
&\quad +\sqrt{N-1}\sqrt{1-b_+^2}(A+\sqrt{N-1}\int|A'| + (N-2)A\Omega T)
\end{aligned}$$

and a point fix study leads to

$$\begin{aligned}
&1-b_+ \leq 2(\mathtt{N}-1)\mathtt{A\Omega T} \qquad (7)\\
&\quad +2(N-1)(A+\sqrt{N-1}\int|A'| + (N-2)A\Omega T)^2.
\end{aligned}$$

Finally one (not optimized) adiabatic condition is

$$A + \sqrt{N}\int|A'| + (\sqrt{\mathtt{N}}+N-2)A\Omega T \ll \frac{1}{\sqrt{N}} \qquad (8)$$

We define two useful *reals* A_{km}:

$$A_{km}^{(1)} = \frac{|\langle m|\dot{k}\rangle|}{i(\langle k|\dot{k}\rangle - \langle m|\dot{m}\rangle) - \frac{E_k-E_m}{\hbar} + \frac{d}{dt}\arg\langle m|\dot{k}\rangle} = \frac{|\langle m|\dot{k}\rangle|}{\dot{\gamma}_{km}^{(1)}}$$

for the $\theta_{km} = \theta_{km}^{(1)}$ choice , and $A_{km}^{(2)} = \frac{|\langle m|\dot{k}\rangle|}{\dot{\gamma}_{km}^{(2)}}$ for the $\theta_{km} = \theta_{km}^{(2)}$ choice where $\dot{\gamma}_{km}^{(2)} = \dot{\gamma}_{km}^{(1)} + \sum_{j\neq m} \frac{|\langle \dot{m}|j\rangle|^2}{\dot{\gamma}_{jm}^{(2)}}$. When the hamiltonian $H(t)$ is real in the canonical basis, the eigenstates $|m\rangle$ and $\langle m|\dot{k}\rangle$ are reals and $\langle m|\dot{m}\rangle = 0$ so, $|A_{km}^{(1)}| = |A_{km}^{(0)}|$.

If all $A_{km}^{(1)}$, or $A_{km}^{(2)}$, are *monotonics* in $[0,T]$ $\int_0^T |\dot{A}_{km}| = |A_{km}(T) - A_{km}(0)|$ and the condition (8) becomes simpler: $A^{(1)} + \sqrt{N}A^{(1)}\Omega T \ll 1/N$ or $A^{(2)} + \sqrt{N-2}A^{(2)}\Omega T \ll 1/N$, where $A^{(i)}$ indicates that it should be calculated using the $A_{km}^{(i)}$ choice. For $N = 2$ smallness and monotonicity of $A_{+-}^{(1)} = \frac{|\Omega_L|}{2\delta_L}$ is equivalent to smallness and no more than one monotonicity change of $A_{+-}^{(2)} = \frac{|\Omega_L|}{\delta_L+\sqrt{\delta_L^2+|\Omega_L|^2}} \geq 0$. Thus, a final general, simple and useful adiabatic condition is (for monotonics $A_{km}^{(1)}$)

$$A^{(1)} + \sqrt{N-2}A^{(1)}\Omega T \ll 1/N. \qquad (9)$$

It is even possible to refine the condition by dividing the interval $[0,T]$ in smaller intervals where all $A_{km}^{(i)}$ are monotonics. A perturbative point of view, neglecting the $A^{(1)}\Omega T$ term, has been used to derive similar results [18].

The $N = 2$ case is illustrative because it is the only one where a time independent adiabatic condition exists:

$$2|A^{(1)}| = \left|\frac{\Omega_L}{\delta_L}\right| \ll \frac{1}{M^2} \qquad (10)$$

where $M-1$ is the number of monotonicity change of $\frac{|\Omega_L|}{\delta_L}$ in $[0,\infty]$. This generalize the Schwinger conditions Eq. (2). For real hamiltonian the condition is

$$2|A^{(1)}| = 2|A^{(0)}| = \left|\frac{\dot{\theta}}{\omega_0}\right| = \left|\frac{\Omega_0\dot{\delta}_0}{(\delta_0^2+\Omega_0^2)^{3/2}}\right| \ll \frac{1}{M^2}$$

and becomes the usual adiabatic condition if M is small, for instance if the matrix elements $\dot{\delta}_0, \Omega_0$ of H and $\dot{H}$ have small number of monotonicity changes. This explain why the real dressed state hamiltonian, $H_0 = R_0^\dagger H R_0 - i\hbar R_0^\dagger \dot{R}_0 = -\frac{\hbar}{2}\left(\begin{smallmatrix}\delta_0 & \Omega_0\\ \Omega_0 & -\delta_0\end{smallmatrix}\right)$, obtained from H in the rotating frame (with the simple phase choice $\theta_+ = \theta_- = 0$) or simply by $\omega_L = 0$, have been *luckily* combined with the usual adiabatic theorem to describe several adiabatic evolutions such as, the RAP (Rapid Adiabatic Passage), the SCRAP (frequency or Stark-Chirped RAP) or the STIRAP (STImulated Raman Adiabatic Passage).

However when real oscillatory terms are present the usual adiabatic condition is no more sufficient to provide adiabatic evolution. As example we use the cycling hamiltonian [19, 20], $H = H_0$ with $\delta_0(t) = \alpha\cos(\omega t)$ and α, ω, Ω_0 are (positives to simplify) constants. It is relevant in many areas in physics: magnetic resonance, atomic collision, laser-atom interactions without the RWA and even localization by exchanging the parameters δ_0 and Ω_0 (hamiltonian $R_y H_0 R_y^\dagger$ with $R_y = e^{i\pi\sigma_y/4}$). The weak-coupling and large amplitude case $\alpha \gg \Omega_0, \omega$ is simple because the non-adiabatic transition probability p_1 (so called single-passage or one-way transition) is given by one of the simplest of the several existing approximate formulas (Landau-Zener-Stückelberg,

Rosen-Zener-Demkov, Nikitin, Zhu-Nakamura models, ... [1, 21]) namely the Landau-Zener one: $p_1 \approx e^{-2\pi\frac{\Omega_0^2}{4\alpha\omega}} = e^{-\pi/(4A^{(1)}(\infty))}$ [22]. The $M = 2$ double-passage transition probability p_2, which depends of a relative (Stückelberg) phase Θ of the wavefunction, $p_2 = 4p_1 \sin^2(\Theta)$ can be 4 times higher than p_1 and the M (even) multiple passage probability $p_M \approx p_1 \frac{\sin^2 M\Theta}{\cos^2 \Theta}$ can be M^2 times higher than p_1. Here small ω value leads to the adiabatic limit $p_1 \to 0$ and with $\Theta \simeq \frac{\alpha}{\omega} \sim \pi/2$ we could have $p_M \sim 1$ [22]. Interestingly enough, the reverse case, namely the diabatic limit ($p_1 \to 1$) can leads (for instance when α/ω annul the Bessel J_0 function) to the reverse phenomenum of adiabaticity created after multiple passages ($p_M \approx 0$) known as suppression of the tunneling, coherent destruction of tunneling, dynamical localization or population trapping depending on the context [19, 22].

This two level example illustrate why monotonicity is require to avoid constructive interferences transforming an adiabatic (resp. diabatic) single passage in a fully diabatic (resp. adiabatic) transition after multiple passages. The two level system with several crossings is very similar to the case of single crossing but with several levels leading to sum of dephased Landau-Dykhne-Davis-Pechukas formulas [23, 24]. Moreover, the transition probability in a multilevel system is the product of several Landau-Dykhne type terms, corresponding to several successive transitions between pairs of levels [25]. However, several consecutive constructive interferences are exceptional and the generic most common case concern a system "complex enough" with small total probability when the single crossing probability is small [26].

In conclusion, we have derived exact bounds for the evolution Eqs. (5), (7) as well as general adiabaticity criterion Eqs. (9), (10). The key parameters for adiabaticity are the smallness and the small number of monotonicity change of $A^{(1)} \sim \frac{1}{\gamma^{(1)}} \frac{\|\dot{H}\|}{\Delta E}$ as well as a short evolution time ($T^{-1} \gg (N-2)^{3/2} \frac{1}{\gamma^{(1)}} \frac{\|\dot{H}\|^2}{\Delta E^2}$). For real hamitonian the adiabatic (Pancharatnam) phase type $\gamma^{(1)}$ is the spectrum frequency gap and the usual adiabatic condition are restored if the matrix elements of H and $\dot{H}$ have small number of monotonicity changes in the two level ($N = 2$) case. The results presented here, and demonstrated for the discrete, but possibly degenerate case, might be useful for adiabatic quantum evolution and adiabatic quantum computation studies. Extension to the infinite dimensional or non hermitian cases are some of the next steps needed to derive more universal quantum adiabatic conditions.

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[27] We use the time derivative of $\langle m|n\rangle$ and $\langle m|H|n\rangle$ leading, for non degenerate case, to $-\langle \dot{m}|n\rangle = \langle m|\dot{n}\rangle = -\frac{\langle m|\dot{H}|n\rangle}{E_m - E_n}$.
[28] An important example is the interpolating hamiltonian $H(t) = H_{\rm in}(1-t/T) + H_{\rm fin}t/T$. $H(t) = \hat{H}(s(t))$ have also been considered with a *monotonic* function $s(t) \in [0,1]$ controlling locally the speed of the process. When the timing T is not an issue $s = t/T$ is the simplest choice.